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Analytical solutions of multiple light scattering problems: a review

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Abstract
Several approximate solutions of the radiative transfer equation are reviewed. They can be used for the retrieval of sizes and chemical compositions of dispersed particles from measurements of transmitted, reflected or internal light fields for the case of plane-parallel slabs containing various substances in a dispersed state (e.g. aerosols, colloids, suspensions, foams and sprays). The optical thickness of a sample can, in principle, be arbitrary.

Keywords: radiative transfer, light scattering, laser diffractometer, laser sizer, environmental monitoring, combustion diagnostics

1. Introduction

Light scattering is a major technique in characterization of particulate systems. Spectral and angular variations of intensity and polarization of scattered light beams depend on the microstructure (concentration, internal structure, size and shape) and chemical composition (via spectral variation of optical constants) of particles. Thus, all major characteristics of disperse systems can, in principle, be obtained from light scattering experiments.

There is no single approach to the solution of a given particle characterization problem. For instance, optical particle sizing techniques vary depending on the parameters of the disperse system in question. Particulate media with volume scattering functions that are strongly elongated in the forward direction are studied with small-angle techniques (Bayvel and Jones 1981). Those with appreciable spectral extinction dependences are investigated with inverse procedures based on spectral extinction measurements (Shifrin and Tonna 1993).

Complications arise in the case of the occurrence of multiple light scattering in a sample (Khlebtsov 1984, Belov *et al* 1984, Vagin and Veretennikov 1989, Hirleman 1991, Zege and Kokhanovsky 1992, Schnablegger and Glatter 1995, Borovoi 1995, 1998, Kokhanovsky 2001). This multiple light scattering can be observed both for highly concentrated suspensions and for disperse systems with small concentrations of particles (typically less than 1% by volume) if the optical thickness of a sample is not small.

Generally speaking, scattering laws for highly concentrated suspensions and diluted disperse systems can differ considerably. There is even a possibility of dispersed layers with high concentrations of particles exhibiting a kind of Bragg maxima (Lock and Chiu 1994). In this case the disperse system can form a completely ordered (or semi-ordered) structures (e.g. bubbles in foam on the surface of water), which has common features with crystalline media. In the case of semi-ordered disperse systems there is a liquid-like behaviour. This can be described in terms of the correlation functions of positions of particles (Ivanov *et al* 1988). Particles can serve as an analogy of molecules in a liquid in this case. This makes it possible to introduce into the scattering optics of disperse systems mathematical notions that have been known for a long time in the theory of liquids and semi-ordered structures (e.g. correlation functions, which describe the probability of particles or pairs of particles being separated by a particular distance). Clearly, such a probability is equal to zero for distances smaller than the diameter of impenetrable or hard particles. The probability goes to unity for a low concentration of particles. In this case we have a chaotic distribution of particles in the scattering volume and there is no preferential distance between particles.

We will consider this last case of multiple light scattering by particles randomly distributed in space in more detail here. This case is relatively easy to study in comparison with the case of highly correlated distributions of particles. The general approach to the problem is well known. It is based on the radiative transfer equation for the Stokes vector of reflected and transmitted light beams (Kokhanovsky 2001). The same method can be used for studies of a light field inside a scattering medium. The coupled system of four integro-differential equations for a given concentration of particles, their size and the thickness of a layer can easily be solved numerically and codes are available via the internet nowadays. For instance, the bidirectional reflection function of plane-parallel semi-infinite media with arbitrary values of the single-scattering albedo and the phase function can be calculated with the set of programs located at http://www.giss.nasa.gov/∼crmim. The code is based on the numerical solution of Ambarzumian's nonlinear integral equation (Mishchenko *et al* 1999). One can find radiative transfer codes for finite slabs at http://atol.ucsd.edu/∼pflatau.

The solution of the direct problem can be applied also for the development of retrieval procedures. For instance, the so-called library method of inverse problem solution is based on the comparison of the pre-calculated sets of the reflection or transmission functions (or matrices in the case of polarized light analysis) with measured data (Nakajima and King 1990). The accuracy of the retrieval procedure will depend on the grid scale in the library of calculated functions and the sensitivity of the measured values to the microstructural parameters to be retrieved.

Yet another approach consists of the application of the approximate solutions of the radiative transfer equation, which often take analytical or semi-analytical form. For their application, the light scattering medium being studied should satisfy given conditions (e.g. small or large optical thickness), which can, in principle, be achieved by changing the concentration of particles or the thickness of the medium artificially. The size of particles can easily be retrieved in the case of optically thin (and, therefore, singly scattering layers).

On the other hand, the large thickness of a disperse medium enhances the sensitivity of the reflected light to the absorption coefficient of particles. This can be used for the development of the spectroscopy of light scattering media (Rozenberg 1967, Dubova *et al* 1977, 1981). Analytical solutions often allow one to reduce the problem of multiple light scattering to the more simple problem of single light scattering. This means that they provide analytical relations between measured global optical characteristics of a medium in question and local optical parameters (e.g. volume scattering and extinction coefficients). In any case, they provide powerful techniques for studies of the information content of measurements. So, it is often, if not always, desirable to study the multiple light scattering medium in question with simple analytical techniques, before going to more advanced and accurate approaches.

The task of this paper is to review analytical solutions of the radiative transfer equation, which are often used in the practice of particulate systems analysis. The topic is very diverse and wide. In principle, it deserves to be considered in detail in a separate book. However, the limited space allocated to papers in this issue allows us to consider only selected results. Thus, we will present here in a short and instructive form the solutions of the radiative transfer equation only in the cases of angular scattering coefficients that are strongly peaked in the forward direction, optically thin and optically thick light scattering layers. Also we will consider only the first element of the Stokes vector, namely the light field intensity. However, most of the results can easily be generalized to account for polarization of light.

2. The radiative transfer equation

Generally speaking, one needs to solve the wave equation for a system of N particles ($N \to \infty$) in an electromagnetic field for studies of optical properties of multiply light scattering media. This is a difficult problem, as can easily be understood from the complexity of the laws governing light scattering in the case of light scattering by a single particle (Bayvel and Jones 1981). However, under some approximations it is possible to derive a simpler radiative transfer equation (RTE) from a general wave equation for N particles in an electromagnetic field (Ishimaru 1978, Apresyan and Kravtsov 1983, Papanicolaou and Burridge 1975):

$$\sigma_{ext}^{-1}(\vec{n}\vec{\nabla})I_t(\vec{r},\vec{n}) = -I_t(\vec{r},\vec{n}) + \frac{\omega_0}{4\pi}\int_{4\pi} p(\vec{n},\vec{n}')I_t(\vec{r},\vec{n}')\,\mathrm{d}\Omega' + B_0(\vec{r},\vec{n}) \quad (1)$$

where ω_0 is the single-scattering albedo, which is equal to the ratio of scattering (σ_{sca}) and extinction (σ_{ext}) coefficients, $p(\vec{n},\vec{n}')$ is the phase function, $\vec{r} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$ is the radius vector of the observation point and the vector $\vec{n} = l\vec{e}_x + m\vec{e}_y + n\vec{e}_z$ determines the direction of a light beam with the intensity I_t. The function $B_0(\vec{r},\vec{n})$ describes the internal sources of radiation. We will assume that $B_0(\vec{r},\vec{n}) = 0$, which is often the case in the optical range of the electromagnetic spectrum.

For a plane parallel light scattering layer, illuminated at every point on the top by a unidirectional beam of light, the radiative transfer equation (1) can be rewritten in the following form (Chandrasekhar 1950, Sobolev 1956, 1972, Van de Hulst 1980):

$$\cos\vartheta\frac{\mathrm{d}I(\tau,\vartheta,\vartheta_0,\phi)}{\mathrm{d}\tau} = -I(\tau,\vartheta,\vartheta_0,\phi) + B(\tau,\vartheta,\vartheta_0,\phi) \quad (2)$$

where

$$B(\tau,\vartheta,\vartheta_0,\phi) = \frac{\omega_0}{4\pi}\int_0^{2\pi}\mathrm{d}\phi'\int_0^{\pi} I(\tau,\vartheta',\vartheta_0,\phi')p(\theta')\sin\vartheta'\,\mathrm{d}\vartheta' + \frac{\omega_0 I_0}{4}p(\theta)\exp\left(-\frac{\tau}{\cos\vartheta_0}\right). \quad (3)$$

Here $\tau = \sigma_{ext}L$ is the optical thickness of a layer, L is the geometrical thickness, ϑ_0 is the angle of incidence, ϑ is the angle of observation, ϕ is the azimuth of the observed radiation, ω_0 is the single-scattering albedo, $p(\theta)$ is the phase function, $I(\tau,\vartheta,\vartheta_0,\phi)$ is the diffused intensity at the optical thickness τ in the direction (ϑ,ϕ), πI_0 is the net flux per unit area normal to the beam and

$$\cos\theta' = \cos\vartheta\cos\vartheta' + \sin\vartheta\sin\vartheta'\cos(\phi-\phi')$$
$$\cos\theta = \cos\vartheta\cos\vartheta_0 + \sin\vartheta\sin\vartheta_0\cos\phi. \quad (4)$$

We have assumed that the azimuth of the incident radiation ϕ_0 is equal to zero. The total intensity I_t (see equation (1)) is

$$I_t = I(\tau, \vartheta, \vartheta_0, \phi) + \pi I_0 \exp\left(-\frac{\tau}{\cos\vartheta_0}\right)\delta(\vec{\Omega} - \vec{\Omega}_0) \quad (5)$$

where $\delta(\vec{\Omega} - \vec{\Omega}_0)$ is the delta function and $\vec{\Omega}_0$ is the solid angle in the direction of incidence. Equation (2) is simpler than equation (1) because it holds only for the diffused light. One can account for the direct light with equation (5). Boundary conditions for equation (2) can be presented in the following form (Sobolev 1972):

$$I(0, \vartheta, \vartheta_0, \varphi) = 0 \qquad \text{at } \vartheta < \pi/2$$
$$I(\tau_0, \vartheta, \vartheta_0, \varphi) = 0 \qquad \text{at } \vartheta > \pi/2. \quad (6)$$

These conditions emphasize that the diffused radiation does not enter a scattering layer, neither from the top ($\tau = 0$) nor from the bottom ($\tau = \tau_0$), in the case being studied.

One can formally solve equation (2) and obtain

$$I(\tau, \eta, \xi, \phi) = \frac{e^{-\tau/\eta}}{\eta}\int_0^{\tau} B(\tau', \eta, \xi, \phi)\, e^{\tau'/\eta}\, d\tau' \qquad \text{at } \eta > 0 \quad (7)$$

and

$$I(\tau, \eta, \xi, \phi) = \frac{e^{-\tau/\eta}}{\eta}\int_{\tau_0}^{\tau} B(\tau', \eta, \xi, \phi)\, e^{\tau'/\eta}\, d\tau' \qquad \text{at } \eta < 0 \quad (8)$$

where $\eta = \cos\vartheta$ and $\xi = \cos\vartheta_0$. Also, we define $\mu = |\eta|$ and $\mu_0 = |\xi|$. It follows from equation (4) that $\cos\theta = (-1)^l \mu\mu_0 + [(1-\mu^2)(1-\mu_0^2)]^{1/2}\cos\phi$, where $l = 1$ for the reflected radiance and $l = 2$ for the transmitted radiance. If the function $B(\tau', \eta, \xi, \phi)$ is known, equations (7) and (8) can be used to calculate the intensity of the light field at any optical thickness τ inside a layer. Equation (7) represents the downward radiation and equation (8) represents the upward radiation. The first boundary condition in equation (6) can be obtained from equation (7) at $\tau = 0$. The second boundary condition follows from equation (8) at $\tau = \tau_0$.

Very often one needs to know the diffused light intensity escaping from the top ($I_\uparrow^d(0, \eta, \xi, \phi)$) and the bottom ($I_\downarrow^d(\tau_0, \eta, \xi, \phi)$) of a scattering layer. They can be obtained from the general equations (7) and (8):

$$I_\uparrow^d(0, \eta, \xi, \phi) = -\frac{1}{\eta}\int_0^{\tau_0} B(\tau', \eta, \xi, \phi)\, e^{\tau'/\eta}\, d\tau' \qquad \eta < 0 \quad (9)$$

and

$$I_\downarrow^d(\tau_0, \eta, \xi, \phi) = \frac{e^{-\tau_0/\eta}}{\eta}\int_0^{\tau_0} B(\tau', \eta, \xi, \varphi)\, e^{\tau'/\eta}\, d\tau' \qquad \eta > 0. \quad (10)$$

The function $B(\tau', \eta, \xi, \varphi)$ is not known *a priori* in most cases. The integral equation for this function can be obtained from equations (3), (7) and (8):

$$B(\tau, \eta, \xi, \phi) = \frac{\omega_0}{4\pi}\int_0^{2\pi} d\phi' \left[\int_0^1 p(\theta')\, d\eta' \int_0^{\tau} B(\tau', \eta', \xi, \phi') \times \exp\left(\frac{\tau'-\tau}{\eta'}\right)\frac{d\tau'}{\eta'} - \int_{-1}^{0} p(\theta')\, d\eta' \int_0^{\tau_0} B(\tau', \eta', \xi, \phi') \times \exp\left(\frac{\tau'-\tau}{\eta'}\right)\frac{d\tau'}{\eta'}\right] + \frac{I_0\omega_0}{4} p(\theta)\, e^{-\tau/\xi}. \quad (11)$$

Equation (11) cannot be solved analytically. Various numerical and approximate methods can be used to solve equations (2) and (7)–(11) (Lenoble 1985).

It is evident that the solution of equation (2) for the diffused intensity should depend on the optical thickness τ_0, single-scattering albedo ω_0, phase function $p(\theta)$, depth τ and geometrical parameters (the cosines η and ξ and the relative azimuth ϕ). This solution is not known in the general case. However, it can be derived approximately for small or large values of the optical thickness τ_0. Note that the accuracy of selected multiple-scattering approximations was studied in detail by King and Harshvardhan (1986).

3. Optically thin layers

For a thin layer ($\tau \to 0$) one can neglect photons scattered more than once (the integral in equation (11) is equal to zero) and obtain the following analytical solution for the source function (see equation (3)):

$$B(\tau, \vartheta, \vartheta_0, \phi) = \frac{I_0\omega_0 p(\theta)}{4} e^{-\tau/\xi}. \quad (12)$$

It follows from equations (9), (10) and (12), under the assumption that the values of ω_0 and $p(\theta)$ do not depend on the depth τ (a homogeneous layer) (Sobolev 1956), that

$$I_\uparrow^d = \frac{\omega_0 I_0 \xi}{4(\mu+\xi)}\left\{1 - \exp\left[-\tau_0\left(\frac{1}{\mu}+\frac{1}{\xi}\right)\right]\right\} p(\theta) \quad (13)$$

$$I_\downarrow^d = \frac{\omega_0 I_0 \xi}{4(\mu-\xi)}\left[\exp\left(-\frac{\tau_0}{\mu}\right) - \exp\left(-\frac{\tau_0}{\xi}\right)\right] p(\theta) \qquad \text{at } \mu \neq \xi \quad (14)$$

and

$$I_\downarrow^d = \frac{\omega_0 I_0 \tau_0}{4\mu} e^{-\tau_0/\mu} p(\theta) \qquad \text{at } \mu = \xi \quad (15)$$

where $I_\downarrow^d(I_\uparrow^d)$ denotes the downward (upward) diffused light intensity. For the reflection function $R = I_\uparrow^d/(\xi I_0)$ and the transmission function $T = I_\downarrow^d/(\xi I_0)$, it follows from equations (13) and (14) that

$$R(\tau_0, \mu, \xi, \phi) = \frac{\omega_0 p(\theta)}{4(\mu+\xi)}\left\{1 - \exp\left[-\left(\frac{1}{\mu}+\frac{1}{\xi}\right)\tau_0\right]\right\} \quad (16)$$

and

$$T(\tau_0, \mu, \xi, \varphi) = \frac{\omega_0 p(\theta)}{4(\mu-\xi)}(e^{-\tau_0/\mu} - e^{-\tau_0/\xi}) \quad (17)$$

where $\theta = \arccos\{\mu\xi + [(1-\mu^2)(1-\xi^2)]^{1/2}\cos\phi\}$ for the transmitted light and $\theta = \arccos\{-\mu\xi + [(1-\mu^2)(1-\xi^2)]^{1/2}\cos\phi\}$ for the reflected light. These two equations are extremely important and have already been used in many applications. It follows for $\tau_0 \to \infty$ from equation (16) that

$$R = \frac{\omega_0 p(\theta)}{4(\mu+\xi)}.$$

This equation represents the contribution of the singly scattered light to the reflection function of a semi-infinite medium. One can see that this contribution is larger for weakly absorbing media ($\omega_0 \approx 1$) and large angles of incidence and observation $\mu \approx \xi \approx 0$ of $\vartheta \approx \vartheta_0 \approx \pi/2$. It depends on the phase function $p(\theta)$ of a scattering medium too.

4. Optically thick layers

4.1. General equations

In many cases the optical thickness of light scattering layers being studied is much larger than 1. Then the asymptotic equations, which also follow from the radiative transfer equation (2) as $\tau \to \infty$ can be used. The mathematical derivation of these equations is quite involved (Kokhanovsky 2001). However, they have quite simple final forms, which can be obtained also from physical arguments (Van de Hulst 1980):

$$R(\tau_0, \mu, \mu_0, \phi) = R_\infty(\mu, \mu_0, \phi) - T(\mu, \mu_0) l \exp(-k\tau_0)$$
$$T(\mu, \mu_0) = tK(\mu)K(\mu_0) \tag{18}$$

where $R_\infty(\mu, \mu_0, \phi)$ is the reflection function of a semi-infinite medium with the same local optical characteristics (ω_0, $p(\theta)$) as the finite layer being studied,

$$t = \frac{m\,\mathrm{e}^{-k\tau_0}}{1 - l^2\,\mathrm{e}^{-2k\tau_0}} \qquad l = 2\int_0^1 K(\mu) i(-\mu)\,\mathrm{d}\mu$$
$$m = 2\int_{-1}^1 i^2(\mu)\mu\,\mathrm{d}\mu. \tag{19}$$

The functions $i(\mu)$ and $R_\infty(\mu, \mu_0, \phi)$ can be derived from the following integral equations (Ambarzumian 1961, Sobolev 1972, Van de Hulst 1980):

$$i(\mu) = \frac{\omega_0}{2(1 - k\mu)} \int_{-1}^1 p(\mu, \mu') i(\mu')\,\mathrm{d}\mu' \tag{20}$$

$$\begin{aligned}
R_\infty(\mu_0, \phi_0, \mu, \phi) &= \frac{\omega_0}{4(\mu + \mu_0)} p(\theta) \\
&+ \frac{\mu_0\omega_0}{4\pi(\mu_0 + \mu)} \int_0^1 \int_0^{2\pi} p(\mu, \phi, \mu', \phi') \\
&\times R_\infty(\mu', \phi', \mu_0, \phi_0)\,\mathrm{d}\mu'\,\mathrm{d}\phi' \\
&+ \frac{\mu\omega_0}{4\pi(\mu_0 + \mu)} \int_0^1 \int_0^{2\pi} p(\mu_0, \phi_0, \mu', \phi') \\
&\times R_\infty(\mu', \phi', \mu, \phi)\,\mathrm{d}\mu'\,\mathrm{d}\phi' \\
&+ \frac{\omega_0\mu\mu_0}{4\pi^2(\mu_0 + \mu)} \int_0^1 \mathrm{d}\phi' \int_0^{2\pi} R_\infty(\mu', \phi', \mu, \phi)\,\mathrm{d}\mu' \\
&\times \int_0^{2\pi} \mathrm{d}\phi'' \int_0^1 p(-\mu', \phi', \mu'', \phi'') \\
&\times R_\infty(\mu'', \phi'', \mu_0, \phi_0)\,\mathrm{d}\mu''.
\end{aligned} \tag{21}$$

Note that the function $i(\mu)$ is the solution of equation (20) at the minimal characteristic number k and (Sobolev 1972)

$$K(\mu) = \frac{i(\mu)}{m} - \frac{2}{m}\int_0^1 R_\infty(\mu, \mu_0) i(-\mu_0)\mu_0\,\mathrm{d}\mu_0 \tag{22}$$

where

$$R_\infty(\mu, \mu_0) = \frac{1}{2\pi}\int_0^{2\pi} R_\infty(\mu, \mu_0, \phi)\,\mathrm{d}\phi. \tag{23}$$

The asymptotic relations (18) have already been used in many applications (Rozenberg 1967, Van de Hulst 1980, King 1987, Minin 1988, Zege *et al* 1991, Kokhanovsky and Zege 1996). It is difficult to apply them directly for the calculation of the reflection and transmission functions of particulate media, because they include two unknown functions, $R_\infty(\mu, \mu_0, \phi)$ and $K(\mu)$, and three unknown constants, k, l and m. However, it is important that these functions do not depend on the optical thickness τ_0. They depend only on the geometry of the problem (e.g. angles of viewing and incidence), single-scattering albedo ω_0 and phase function. Note that these auxiliary functions can be calculated, e.g. with codes developed by Konovalov (1975) and Nakajima and King (1992).

Equations (18) can be used to generate simple approximations in some particular cases. Here we consider two of them: nonabsorbing media ($\omega_0 = 1$) and weakly absorbing media ($\omega_0 \to 1$).

4.2. Nonabsorbing optically thick media

Let us consider now the case of nonabsorbing media in more detail. It follows from equation (18) at $\omega_0 = 1$ (Van de Hulst 1980) that

$$R(\tau, \mu, \mu_0, \phi) = R^0_\infty(\mu, \mu_0, \phi) - T(\mu, \mu_0)$$
$$T(\mu, \mu_0) = \frac{K_0(\mu)K_0(\mu_0)}{0.75(1 - g)\tau_0 + \Delta} \tag{24}$$

where g is the asymmetry parameter,

$$\Delta = 3\int_0^1 K_0(\mu)\mu^2\,\mathrm{d}\mu \tag{25}$$

and $K_0(\mu)$ denotes the function $K(\mu)$ at $\omega_0 = 1$. $R^0_\infty(\mu, \mu_0, \phi)$ is the reflection function of a semi-infinite cloud at $\omega_0 = 1$. Note that values of $K_0(\mu)$ and $R^0_\infty(\mu, \xi, \phi)$ depend on the phase function of a scattering medium only. They do not depend on the optical thickness. Moreover, even the dependences of these functions on the phase function are rather weak. It was shown (Zege *et al* 1991) that, with an error of less than 2%,

$$K_0(\mu) = \frac{3}{7}(1 + 2\mu) \tag{26}$$

for $\mu > 0.2$. Thus, it follows from equations (25) and (26) that $\Delta = 15/14$. It is important that this simple expression for the escape function is consistent with the exact normalization condition (Sobolev 1972)

$$2\int_0^1 K_0(\mu)\mu\,\mathrm{d}\mu = 1. \tag{27}$$

It follows for the second moment in the same approximation that

$$\int_0^1 K_0(\mu)\mu^2\,\mathrm{d}\mu = Q \tag{28}$$

where $Q = \frac{5}{14}$. To apply equations (24) for the solution of a specific light scattering problem one should know also the function $R^0_\infty(\mu, \mu_0, \phi)$.

Unfortunately, there is no simple expression for this function. The reflection $R^0_\infty(\mu, \mu_0, \phi)$ function depends on the phase function and angles ϑ, ϑ_0 and ϕ. One can assume that this function can be represented as a combination of two terms: the first does not depend on the phase function and the other term strongly depends on the phase function. It is reasonable to assume that the first term is proportional to

$$R^{ms}_\infty(\mu, \mu_0) = \frac{A + B\mu\mu_0 + C(\mu + \mu_0)}{\mu + \mu_0} \tag{29}$$

which is similar to the case of the isotropic scattering (Chandrasekhar 1950, Sobolev 1972). Indeed, one can expect that the multiple scattering term, given by equation (29), depends only weakly on the phase function due to the randomization of directions of motion of photons during their diffusion in a disperse medium. Thus, the approximation of isotropic scattering is quite appropriate in this case. The second term is proportional to the contribution of the single scattering (Chandrasekhar 1950):

$$R_\infty^{SS} = \frac{p(\theta)}{4(\mu + \mu_0)}. \tag{30}$$

As one can see this contribution is directly proportional to the phase function of a scattering medium. Summing up, it follows that

$$R_\infty^0(\mu, \mu_0, \phi) = \frac{A + B\mu\mu_0 + C(\mu + \mu_0)}{\mu + \mu_0} + \frac{p(\theta)}{4(\mu + \mu_0)} \tag{31}$$

where A, B and C are unknown constants. They can be obtained by fitting results calculated from equation (31) to those computed with the exact radiative transfer code. For instance, it follows for water clouds that $A \approx 0.37$, $B \approx 1.94$ and $C = 0$ at nadir observation ($\mu = 1$). The accuracy of equation (31) is better than to within 2% at $\mu = 1$ for angles of incidence less than 85° in this particular case of cloudy media.

4.3. Weakly absorbing optically thick media

The general equation (18) can be simplified not only in the case of nonabsorbing media but also for the case of weakly absorbing light scattering media ($\omega_0 \to 1$). It follows for $\omega_0 \to 1$ (Van de Hulst 1980, Minin 1988, Zege *et al* 1991) that

$$k = [3(1 - \omega_0)(1 - \omega_0 g)]^{1/2} + O(1 - \omega_0) \tag{32}$$

$$l = 1 - 2q_0 k + 2q_0^2 k^2 + O(k^3) \tag{33}$$

$$m = \frac{8k}{3(1 - g)} + O(k^3) \tag{34}$$

$$K(\mu) = K_0(\mu)(1 - q_0 k) + O(k^2) \tag{35}$$

$$R_\infty(\mu, \mu_0, \phi) = R_\infty^0(\mu, \mu_0, \phi) - \frac{4k}{3(1 - g)} K_0(\mu) K_0(\xi) + O(k^2) \tag{36}$$

$$i(\mu) = 1 + \frac{k\mu}{1 - g} \qquad g = \frac{1}{4}\int_{-1}^{1} p(\theta) \sin 2\theta \, d\theta$$

and $q_0 = 2\Delta/[3(1 - g)]$, where Δ is defined by equation (25). One can find the next terms of these expansions in the book written by Minin (1988). Results of numerical computations of these values for any value of ω_0 were published by Yanovitskij (1997). Analytical results for the functions $R_\infty^0(\mu, \mu_0, \phi)$ and $K_0(\mu)$ were presented in the previous section.

Unfortunately, equations (32)–(36) can be used only for very small $\beta = 1 - \omega_0$. However, for many applications (e.g. snow, cloud and foam optics) it is important to have simple formulae which can be applied for larger values of β as well. To derive such equations we note that it follows from equations (34) and (35) that

$$mK(\mu)K(\mu_0) \approx mK_0(\mu)K_0(\mu_0) \tag{37}$$

which is true to order $O(k^2)$.

Finally, equation (18) can be written in the following form:

$$R(\tau_0, \mu, \mu_0, \phi) = R_\infty(\mu, \mu_0, \phi) - T(\tau_0, \mu, \mu_0) l \exp(-k\tau_0) \tag{38}$$

$$T(\tau_0, \mu, \mu_0) = tK_0(\mu)K_0(\mu_0) \tag{39}$$

where

$$t = \frac{m\, e^{-k\tau_0}}{1 - l^2 e^{-2k\tau_0}}. \tag{40}$$

Equation (40) determines the global transmittance

$$t = 2\int_0^1 \mu \, d\mu \int_0^1 \mu_0 \, d\mu_0 T(\tau_0, \mu, \mu_0). \tag{41}$$

The spherical albedo

$$r = \frac{2}{\pi}\int_0^{2\pi} d\phi \int_0^1 \mu \, d\mu \int_0^1 \mu_0 \, d\mu_0 R(\tau, \mu, \mu_0, \phi) \tag{42}$$

can be derived from the following formula (see equation (38)):

$$r = r_\infty - tl \exp(-k\tau_0) \tag{43}$$

where

$$r_\infty = \frac{2}{\pi}\int_0^{2\pi} d\phi \int_0^1 \mu \, d\mu \int_0^1 \mu_0 \, d\mu_0 R_\infty(\tau, \mu, \mu_0, \phi).$$

Approximate formulae for r_∞, l and m in equations (40) and (43) can be found by the comparison of equations (40) and (43) with well known approximate equations (Rozenberg 1962, 1967, Kokhanovsky 2001):

$$r = \frac{\sinh(x)}{\sinh(x + y)} \qquad t = \frac{\sinh(y)}{\sinh(x + y)} \tag{44}$$

where $x = k\tau_0$, $y = 4kq$ and $q = 1/[3(1 - g)]$. Thus, it follows that $l = e^{-y}$ and $m = 1 - l^2$ or $l = r_\infty$ and $m = 1 - r_\infty^2$.

One can obtain approximately (Zege *et al* 1991)

$$R_\infty(\mu, \mu_0, \phi) = R_\infty^0(\mu, \mu_0, \phi) \exp(-yD(\mu, \mu_0, \phi)) \tag{45}$$

where $D(\mu, \mu_0, \phi) = K_0(\mu)K_0(\mu_0)/R_\infty^0(\mu, \mu_0, \phi)$. This equation coincides with the exact result (36) as $\omega_0 \to 1$. It can be used at $\omega_0 \geqslant 0.95$, which is not the case for equation (36).

Thus, equations (38) and (39) can be written in the following simpler forms:

$$R(\mu, \mu_0, \phi) = R_\infty^0(\mu, \mu_0, \phi) \exp(-yD(\mu, \mu_0, \phi)) - T(\mu, \mu_0) e^{-y-x} \tag{46}$$

$$T(\mu, \mu_0) = \frac{\sinh y}{\sinh(x + y)} K_0(\mu)K_0(\mu_0). \tag{47}$$

The application of these important equations to the optical sizing problems was studied in detail by Kokhanovsky and Zege (1996).

5. The quasi-ballistic regime

Let us consider now the case of highly anisotropically light scattering layers (the asymmetry parameter $g \approx 1$) illuminated along the normal. Examples of such media are oceanic waters, bio-liquids and tissues. An approximation studied here is

usually used in the case of coarsely dispersed light scattering media with large particles (radii $a \gg \lambda$, where λ is the wavelength of an incident light beam).

Also we will assume that the optical thickness is not very high (typically less than 5). In this so-called quasi-ballistic regime most of the photons are scattered within the small-angle-scattering region. Again there is a possibility of simplifying the radiative transfer equation. For this, one can assume that $\cos\vartheta = 1$ in equation (2) and obtain (Dolin 1964, Borovoi 1982, Dolin and Levin 1991, Zege *et al* 1991, Alexandrov *et al* 1993, Kokhanovsky 2001)

$$\frac{\mathrm{d}I(\tau,\mu)}{\mathrm{d}\tau} = -I(\tau,\mu) + \frac{\omega_0}{2}\int_0^1 \mathrm{d}\mu' I(\tau,\mu')p(\mu,\mu') \quad (48)$$

where

$$p(\mu,\mu') = \frac{1}{2\pi}\int_0^{2\pi} p(\mu,\mu',\phi)\,\mathrm{d}\phi. \quad (49)$$

We used the fact that the intensity of the scattered light field for layers with randomly oriented particles does not depend on the azimuth for the illumination of a layer along the normal. Note that the value of $I(\tau,\mu)$ is the total intensity (not the diffused intensity as in equation (2)) and it includes the direct light. The phase function $p(\mu,\mu',\varphi)$ in equation (49) can be represented in the following form (Minin 1988):

$$p(\mu,\mu',\phi) = p(\mu,\mu') + 2\sum_{m=1}^{\infty}\cos[m(\phi-\phi')]\sum_{i=m}^{\infty} c_i^m P_i^m(\mu)P_i^m(\mu')$$

where $P_i(\mu)$ and $P_i^m(\theta)$ are Legendre and associated Legendre polynomials, respectively, and

$$p(\mu,\mu') = \sum_{i=0}^{\infty} x_i P_i(\mu)P_i(\mu')$$

$$x_i = \frac{2i+1}{2}\int p(\theta)P_i(\theta)\sin\theta\,\mathrm{d}\theta \qquad c_i^m = x_i\frac{(i-m)!}{(i+m)!}. \quad (50)$$

We will seek the solution of equation (48) in the following form (Zege *et al* 1991):

$$I(\tau,\mu) = \sum_{i=0}^{\infty} b_i(\tau)P_i(\mu). \quad (51)$$

One can obtain from equations (48) and (51)

$$\frac{\mathrm{d}b_i(\tau)}{\mathrm{d}\tau} = -c_i b_i(\tau) \quad (52)$$

where

$$c_i = 1 - \omega_0\frac{x_i}{2i+1} \quad (53)$$

and the orthogonality of Legendre polynomials was used. Thus, it follows (Zege *et al* 1991) that

$$b_i(\tau) = A_i\exp(-c_i\tau) \quad (54)$$

where A_i = constant.

The intensity of the transmitted light can be found with the following formula:

$$I(\tau,\mu) = \sum_{i=0}^{\infty} A_i\,\mathrm{e}^{-c_i\tau}P_i(\mu). \quad (55)$$

Values of A_i can be derived from initial conditions. Let us show this. We will assume that

$$I(0,\mu_0) = I_0\delta(1-\mu_0) \quad (56)$$

where $\delta(1-\mu_0)$ is the delta function and I_0 is the density of the incident light flux. Also we have

$$\delta(1-\mu_0) = \frac{1}{4\pi}\sum_{i=0}^{\infty}(2i+1)P_i(\mu_0). \quad (57)$$

Therefore, it follows from equations (55) and (57) that

$$A_i = \frac{2i+1}{4\pi} \quad (58)$$

and, finally,

$$I(\tau,\mu) = B\sum_{i=0}^{\infty}\frac{2i+1}{2}\,\mathrm{e}^{-c_i\tau}P_i(\mu) \quad (59)$$

where $B = I_0/(2\pi)$. This is the solution of the problem under consideration. Equation (59) describes the angular distribution of the transmitted light for normal incidence. This formula was applied to the problem of optical particle sizing by Schnablegger and Glatter (1995).

Equation (59) can be rewritten in the integral form (Kokhanovsky 2001). Indeed, the phase function $p(\theta)$ has a sharp peak in the forward-scattering direction ($\theta = 0$) and the main contribution to the integral (50) for x_i arises at small scattering angles. Thus, it follows from equation (50) that

$$x_i = \frac{2i+1}{2}\int_0^{\infty} p(\theta)J_0\left(\theta\left(i+\frac{1}{2}\right)\right)\theta\,\mathrm{d}\theta \quad (60)$$

where the asymptotic relationship

$$\lim_{\theta\to 0} P_i(\cos\theta) = J_0\left(\theta\left(i+\frac{1}{2}\right)\right) \quad (61)$$

was used. It follows from equations (59) and (61) and the Euler sum formula

$$\sum_{i=0}^{\infty} f\left(i+\frac{1}{2}\right) \approx \int_0^{\infty} f(\sigma)\,\mathrm{d}\sigma \quad (62)$$

that

$$I(\tau,\vartheta) = \frac{I_0}{2\pi}\int_0^{\infty}\mathrm{d}\sigma\,J_0(\sigma\vartheta)\exp[-\tau(1-\omega_0 P(\sigma))] \quad (63)$$

where

$$P(\sigma) = \frac{1}{2}\int_0^{\infty} p(\theta)J_0(\theta\sigma)\theta\,\mathrm{d}\theta \quad (64)$$

is the Fourier–Bessel transform of the phase function.

Note that equation (63) is easier to handle than equation (59). The integral (64) can be found analytically for special types of phase functions $p(\theta)$ (see table 1).

The integral (63) includes both diffusely transmitted and direct light. The intensity of the direct light is given by the following equation: $I' = I_0\delta(\vartheta)\exp(-\tau)$. Subtracting the direct light from equation (63), we have

$$I^d(\tau,\vartheta) = \frac{I_0}{2\pi}\int_0^{\infty}(\mathrm{e}^{-\tau(1-\omega_0 P(\sigma))} - \mathrm{e}^{-\tau})J_0(\sigma\vartheta)\sigma\,\mathrm{d}\sigma \quad (65)$$

Table 1. Phase functions $p(\theta)$ and their Fourier–Bessel transforms $P(\sigma)$ (Υ is the normalization constant, $x = 2\pi a/\lambda$ is the size parameter) (Kokhanovsky 2001).

$p(\theta)$	$P(\sigma)$
$\dfrac{2\Upsilon \exp(-\Upsilon\theta)}{\theta}$	$\dfrac{\Upsilon}{(r^2+\sigma^2)^{1/2}}$
$2\Upsilon^2 \exp(-\Upsilon\theta)$	$\dfrac{\Upsilon^3}{(\Upsilon^2+\sigma^2)^{3/2}}$
$\dfrac{2}{\Upsilon^2}\exp\left(-\dfrac{\theta^2}{2\Upsilon^2}\right)$	$\exp\left(-\dfrac{\Upsilon^2\sigma^2}{2}\right)$
$\dfrac{4J_1^2(\theta x)}{\theta^2}$	$\begin{cases} \dfrac{2}{\pi}\left\{\arccos\left(\dfrac{\sigma}{2x}\right) - \dfrac{\sigma}{2x}\left[1-\left(\dfrac{\sigma}{2x}\right)^2\right]^{1/2}\right\} & \sigma \leqslant 2x \\ 0 & \sigma > 2x \end{cases}$

where we used the identity

$$\delta(\vartheta) = \frac{1}{2\pi}\int_0^\infty J_0(\sigma\vartheta)\sigma\,\mathrm{d}\sigma. \tag{66}$$

Equation (65) is of importance for the solution both of direct and of inverse problems of light scattering media optics. It is valid at $\tau \leqslant 5$–7 (Vagin and Veretennikov 1989, Kokhanovsky 2001).

6. Conclusion

We have considered here selected approximate solutions of the transport equation (2). They have already been used for the development of the optical particle sizing and particle characterization algorithms, which account for multiple light scattering. Clearly, the equations presented can be applied also for studies of the change in information content due to multiple scattering of photons in a disperse medium.

In particular, Belov *et al* (1984) and Vagin and Veretennikov (1989) used the inversion of equation (65) to obtain information on particle size distributions contained in the Fourier–Bessel spectrum of the phase function $P(\sigma)$. Schnablegger and Glatter (1995) used equation (59) for the same purpose. A similar approach was developed by Zege and Kokhanovsky (1992) and Zuev *et al* (1997) on the basis of the image transfer theory. Borovoi (1995 1998) proposed schemes for the determination of particle size distributions from analysis of backscattered laser beams. They are based on the 'multiple small-scattering plus single backward scattering' approximation (Katsev *et al* 1998).

Equations (24), (38), (39), (46) and (47) were used for the retrieval of the optical thickness and size of particles in clouds and snow by King(1987), Kokhanovsky and Zege (1996) and Zege *et al* (1998). They can be also used for the retrieval of the absorption coefficient of dispersed substances (Dubova *et al* 1977 1981, Kokhanovsky 2001). This is an important issue in modern spectroscopy of light scattering media.

The application of equation (16) to the retrieval of the optical thickness of an aerosol was given by Kokhanovsky (1998). Note that equation (16) can easily be generalized to account for the polarization of light (Hansen and Travis 1974) and the possible optical activity of particles (Kokhanovsky 1999).

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